Consecutive sums

NMCC

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MINERVASKOLAN

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1 Introduction

In this report, written by class 8d Minervaskolan in Umeå, you will read about our work with the assignment about consecutive sums – an area we knew nothing about before this competition. In the first part, you can read about our interpretation of the problem and our thoughts, before we even started to solve the problem. In the second part, you will read about our working process, which will show you how we have worked with these problems and how we overcame the challenges. After that, we will introduce you to our solutions and our own interesting problem. Finally, you will get to read our conclusion and what we have learned from this project.

Enjoy!
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2 Interpretation

2.1 Interpretation of the rules of the assignment

A consecutive sum is the sum of a consecutive sequence of numbers. These are the rules regarding the different ways in which a consecutive sum can be written, according to how we interpreted the assignment:

§1. A consecutive sum is the sum of an arithmetic progression. An arithmetic progression is a series of numbers with the same difference between adjacent numbers. In this case the difference shall be 1. Therefore, the series "1+2+4+5" is not valid since 2-1=1; 4-2=2 and 1≠2. The series "1+2+3+4", on the other hand, is a valid series of numbers.

§2. The arithmetic progression needs to contain more than one term, which is quite logical, since there has to be multiple terms for something to be called a sum.

§3. All the terms in the series have to be integers. This means that "2.5+3.5+4.5" is not a valid series, since none of the terms are integers.

§4. All numbers in the arithmetic progression have to be positive integers. This means that progressions like "-1+0+1+2" are not valid since 0 and (-1) are not positive.

These rules apply to consecutive progressions. The sum of a progression that follows these rules is what we call a consecutive sum. When referring to these rules later in the text, the designation §1–4 will be used.

2.2 Interpretation of the assignment

The three questions we were supposed to work with and answer were:

1. What numbers that can be written as the sum of consecutive numbers?
2. What numbers that cannot be written as a sum of consecutive numbers?
3. What numbers that can be written as a consecutive sum in more ways than one?

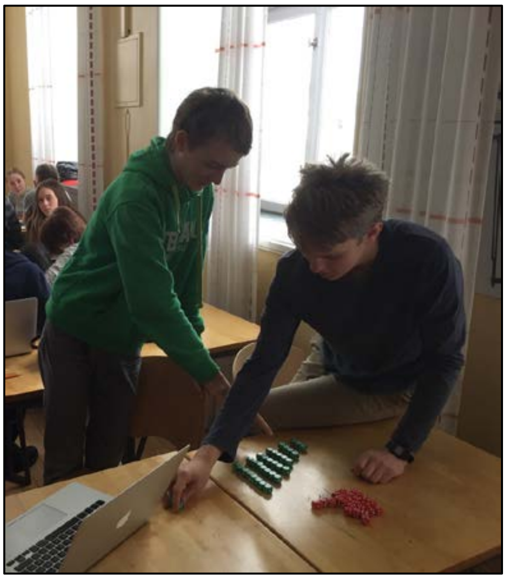
We also chose to make our own question, but more on that later.

When working with the first question, we figured that we were supposed to find some sort of rule for what is required for a number to be able to be written as a consecutive sum.

We considered the second question to be an inverted version of the first question. Thus, to solve this question, all we needed was to find which numbers that did not apply to the rule we were supposed to find.

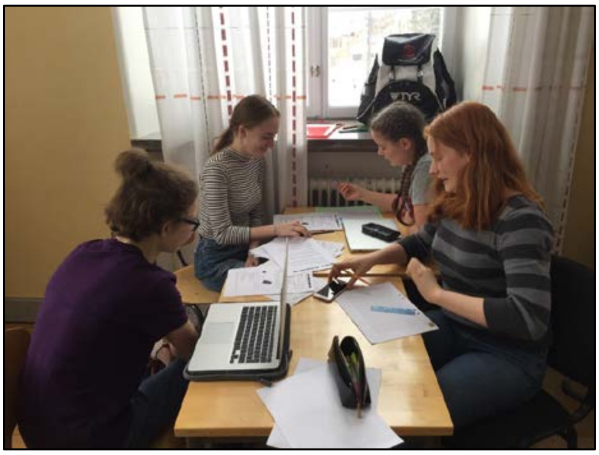
The third question seemed to be the hardest one. We were supposed to build on the rule that we are supposed to define to determine exactly in how many ways a number could be written as a consecutive sum.

3 The working process



We started working on this project by dividing ourselves into several smaller groups, to make the work more efficient. We also came up with methods to simplify the mathematical process which included practical work involving objects like dice (see pictures to the right). Our work in general during the lessons can be described as focused and productive.

To begin the mathematical process, we discussed consecutive sums to get a better understanding of the assignment. Progress in solving the problem was made when the groups tried to determine which numbers could be written as consecutive sums and then tried to see connections and patterns between the numbers. During the process, we had the possibility to use a few different practical tools to simplify the work, for example programming. The programming languages Python and C++ have been used to simplify the explanation of the problem.



Lack of time was one of our biggest challenges. We solved this by working in our spare time and during other lessons that focused on mathematics, for example our science lessons. By dividing ourselves into groups we also made the work more efficient. A few communication errors within the groups were solved by our teacher, who helped us with organizing and planning the work.

4 Our solutions

4.1 Question 1

First, we chose to put numbers in different categories to make it easier to find if they could be written as a consecutive sum.

4.1.1 Odd numbers

All odd numbers can be written in the form of “2n+1” where “n” is an integer. This can be rewritten as “n+(n+1)” which clearly is a consecutive sum. However, observe that odd numbers smaller than 3 cannot be written this way, per §4, which says that no term should be smaller than 1 in the summation. This solution is quite limited, and it only works for some numbers so we need a more general solution.

4.1.2 Even numbers

To find a rule for what even numbers could be written as a consecutive sum, we needed to think in another way. What decides if a number can be written as a consecutive sum? We found that it depends on the numbers odd factors. If the number “(t)” has an odd factor “(2n+1)” we can represent it as:

a x (2n+1) = t

This can be rewritten as:

t = a+a+...+a+a, 2n+1 terms

= (a-n) + (a-(n+1)) +...+ (a-1) +a+ (a+1) +...+ (a+(n+1)) + (a+n), still 2n+1 terms

Because of the fact that “|a+n| > |a-n|”, both “a” and “n” is positive, every eventual negative term will be cancelled out by some of the negative, which means that the final sum may not have 2n+1 terms. The reason for the factor to be odd is that it is impossible to make an arithmetic (§1) series in the beginning with an even number of summands.

4.1.3 All numbers

As shown above, we found that every number with at least one odd factor greater than 1, per §2 (the series cannot have just one term), can be written as a consecutive sum

4.2 Question 2

Because question 2 is the inverse of question 1, we only need to find the numbers that do not follow our rule. According to the rule, only numbers with at least one odd factor greater than 1 can be written as a consecutive sum. It means that numbers with no odd factors cannot be written in this way. All numbers are built up by their prime factors. A number with only even factors only has even prime factors. When the only even prime number is 2, just powers of 2, and as previously stated 1, cannot be written as a consecutive sum.

4.3 Question 3

To solve the third question, we needed to go back and analyze our relationship between a number's odd factors and its ability to be written as a consecutive sum. We found that numbers can be written in that way the same amount of times as the number of odd factors, except for 1, it has. Then it is easily understood that every factor gives a unique sum because it already gives a unique combination of the variables "n" and "a". This means that a number can be written as a consecutive sum in a number of ways that is exactly the number of odd factors it has; which is our final and definitive rule.

4.4 Question 4 – our own question

In our class, there is a group of people that is very interested in geometry. They wondered:

- How can we show this with consecutive sums in a more geometrically and visible way?
- Can we then find some information about this, for example: the greatest and lowest terms in the sum?
- And finally, can we, using information, write the sum in another way, perhaps using the sum symbol?

Per our previous rule, a number with odd factors can be represented as a consecutive sum. A number "(t)" with an odd factor "(2n+1)" can be written as:

$(2n+1) \times a = t$

Using this information, we can show “t” as a rectangle in the same way as we did in early school.

Example using the number 15:

15 = 5 x 3 = (2 x 2+1) x 3

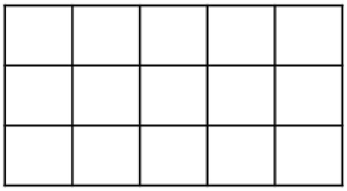


Figure 1

15 = 3 x 5 = (2 x 1+1) x 5

15 = 15 x 1 = (2 x 7+1) x 1

To illustrate this more generally, we chose to split up numbers into two categories. Category 1 consists of sums where “n ≥ a” and category 2 of sums where “a ≤ n”. Here one can see that there are sums who fit in both categories. They are consecutive sums of numbers called triangular numbers.

4.4.1 Category 1

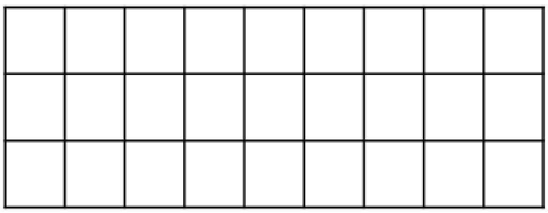


Figure 2

The rectangle is “2n+1” squares long, “a” squares high and it has an area of “2n+1 x a = t”.

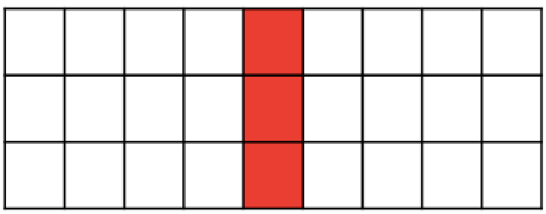


Figure 3

Let us put some colour to the middle term in the sum

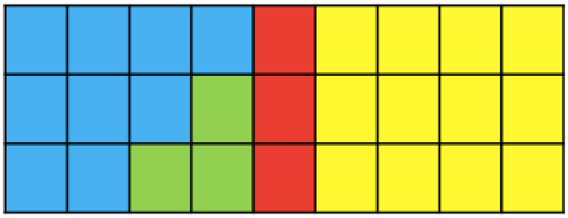


Figure 4

Then we cut through the rectangle from the middle column to the bottom with a slope of -1.

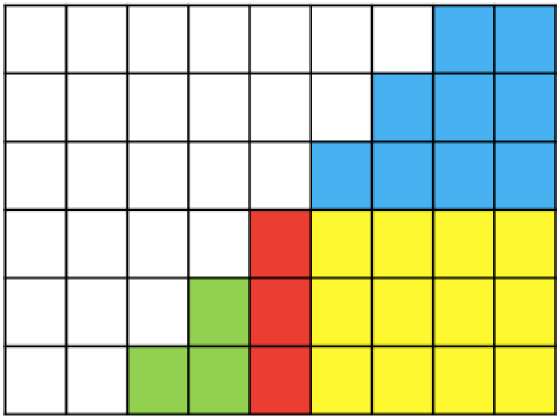


Figure 5

Finally, we move the blue part from figure 4and put it on the top.

To get the consecutive sum in category 1 we need to count the number of squares at every row from up to down. To then find the maximum and minimum worth, we can calculate some lengths in figure 4. Because the green part forms a triangle we find that the maximum is:

(a-1)+n+1 = a+n

The minimum is then:

(2n+1) - ((a-1)+n+1) = 2n+1-a+1-n-1 = n-a+1

This means that:

$$t = \sum_{k=n-a+1}^{n+a} k$$

4.4.2 Category 2

Category 2 is almost the same as category 1. The first difference is in the step where one cuts through the rectangle, the figure then looks like figure 6.

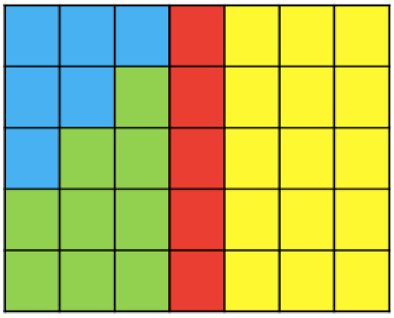


Figure 6

In this category one does not cut through the whole rectangle. However, if we move the blue part like in the previous category, we get figure 7.

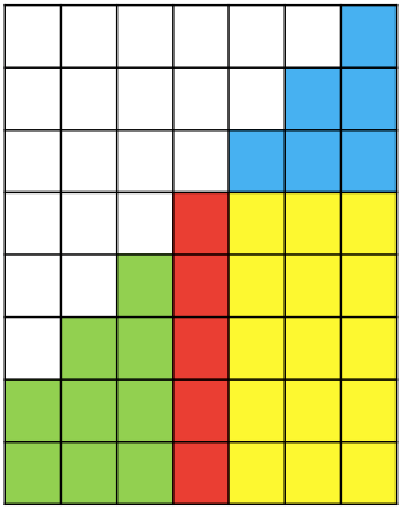


Figure 7

In category 2 we find the consecutive sum in a more natural way when we count the number of squares in every column from left to right. To calculate the maximum and minimum we find the height of the two outer columns. When the blue squares form a rectangle, we get that the left column is “a-n” squares high and the right is “n+a” squares high.

This gives us:

$$t = \sum_{k=a-n}^{n+a} k$$

5 Conclusion

To summarize, the work with the assignment has gone relatively well. We have compiled expressions over which numbers that can be written as consecutive sums. Because no one had any prior knowledge about consecutive sums, we did not make a hypothesis about which answers we would get on the different questions.

At first, several groups experienced that the assignment was easy, but it was perceived as more complicated as the project progressed and the groups' need of other methods grew. We finished the project by discussing results and solutions together. The comparisons were interesting because of the different answers the groups had come up with. The discussions led in turn to each group improving their solutions and conclusions.

This assignment has given us a deeper comprehension of the terms "consecutive numbers" and "consecutive sums". The way that most of the groups solved the assignments has deepened our knowledge about how to discover associations and patterns in different series of numbers; in this case, series of numbers that consist of consecutive sums. The organization of the work has meant a lot of practice when it comes to communicating between each other, not just about mathematics, which means that it has tested our ability to cooperate in our class in general.

Lastly, it is worth mentioning that most of the groups thought that their interpretation of the assignment matched their results, and that the work with the assignment clarified a connection between them both. The solution to the assignment occurred to us when we discovered different patterns for series of consecutive numbers. We also found a rule that says that a number can be written as a consecutive sum in an amount of ways that is exactly the number of odd factors it has (with the exception of 1).